ISOMETRIES OF REAL NORMED VECTOR SPACES

KEATON QUINN

If V is a normed vector space then we turn V into a metric space by defining $d(u,v) = \|u - v\|$. If W another normed vector space we say a function $f : V \to W$ is an *isometry* if $d_W(f(u), f(v)) = d_V(u,v)$ for all $u, v \in V$, or equivalently, if $\|f(u) - f(v)\|_W = \|u - v\|_V$. If f is linear and an isometry then, we have by taking $v = 0$ that $\|f(u)\|_W = \|u\|_V$. This is actually sufficient, take $u = v - w$, then $\|f(v) - f(w)\|_W = \|f(v - w)\|_W = \|v - w\|_V$ so that f is an isometry.

This is not to say that every linear map is an isometry, since for example $f(v) = 2v$ scales length by 2. Nor is this to say that every isometry is linear, take for example, translations, which give $d(u+w, v+w) = \|u + w - (v + w)\| = \|u - v\| = d(u,v)$. However, if our isometry is surjective (and hence bijective since isometries are always injective) and both our vector spaces real, then we know that f is a combination of these two maps. That is, f is a linear transformation followed by a translation: $f = A + w$ for some linear map A and vector w. The original proof of this fact is due to Mazur and Ulam. The proof as presented here is by Nica in [Nic12].

Theorem (Mazur–Ulam, 1932)**.** *Let V and W be two real normed vector spaces and suppose $f : V \to W$ is a surjective isometry, then f is an affine transformation. That is, there exists a linear map $A : V \to W$ and a vector $w \in W$ such that $f = A + w$.*

Proof. Being affine has an equivalent characterization in terms of preserving line segments (we will show the useful direction below). So one of our goals is to show

$$f(ta + (1-t)b) = tf(a) + (1-t)f(b) \tag{$*$}$$

for all a and b. It turns out, preserving line segments will follow from preserving midpoints of line segments. That is, if we have

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2},$$

then by replacing a or b with $\frac{a+b}{2}$ and repeating we get $(*)$ for all $t = \frac{k}{2^n} \in [0,1]$. Since these points are dense in $[0,1]$ by continuity of f we get $(*)$ for all t. We do this now.

Fix two points $a, b \in V$, we show that

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2}$$

for every surjective isometry $f : X \to Y$. To do this we define the possible affine defect f might have as

$$D(f) = \left\| f\left(\frac{a+b}{2}\right) - \frac{f(a) + f(b)}{2} \right\|.$$

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We want to show $D(f) = 0$ for all desired f.

Note that we have a uniform bound on the defect:

$$\begin{aligned} D(f) &\leq \frac{1}{2}\left\|f\left(\frac{a+b}{2}\right) - f(a)\right\|_W + \frac{1}{2}\left\|f\left(\frac{a+b}{2}\right) - f(b)\right\|_W \\ &= \frac{1}{2}\left\|\frac{a+b}{2} - a\right\|_V + \frac{1}{2}\left\|\frac{a+b}{2} - b\right\|_V \\ &= \frac{1}{4}\|a-b\|_V + \frac{1}{4}\|a-b\|_V \\ &= \frac{1}{2}\|a-b\|_V. \end{aligned}$$

It is this bound that helps us get a contradiction to possible positive affine defect. Define ρ as reflection about the vector $\frac{f(a)+f(b)}{2}$ in W, that is

$$\rho : W \to W \quad \text{by} \quad \rho(w) = f(a) + f(b) - w.$$

Now, since f is invertible, define $R(f) : V \to V$ by $R(f) = f^{-1} \circ \rho \circ f$. Note that $R(f)(a) = b$ and $R(f)(b) = a$. Now since f^{-1} is also an isometry, we can compute the affine defect of $R(f)$:

$$\begin{aligned} D(R(f)) &= \left\|R(f)\left(\frac{a+b}{2}\right) - \frac{R(f)(a) + R(f)(b)}{2}\right\|_V \\ &= \left\|f^{-1}\left(\rho\left(f\left(\frac{a+b}{2}\right)\right)\right) - f^{-1}\left(f\left(\frac{a+b}{2}\right)\right)\right\|_V \\ &= \left\|\rho\left(f\left(\frac{a+b}{2}\right)\right) - f\left(\frac{a+b}{2}\right)\right\|_W \\ &= \left\|f(a) - f(b) - f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)\right\|_W \\ &= \left\|2\frac{f(a)+f(b)}{2} - 2f\left(\frac{a+b}{2}\right)\right\|_W \\ &= 2D(f) \end{aligned}$$

Hence, if we iterate this process, we get $D(R^n(f)) = 2^n D(f)$. For large enough n this violates our uniform bound. Thus $D(f) = 0$. Note that is reasoning works for any two $a, b \in V$. We will use this to show that f is affine.

Now, we wish to show f is affine, that is $f = A + w$ for some vector $w \in W$. A quick evaluation at 0 gives $w = f(0)$. So, it suffices to show $f - f(0)$ is linear. Call this function A. We compute for $t \in [0, 1]$,

$$\begin{aligned} A(tv) &= f(tv + (1-t)\cdot 0) - f(0) \\ &= tf(v) + (1-t)f(0) - f(0) \\ &= tf(v) - tf(0) \\ &= tA(v). \end{aligned}$$

We will get the remaining t in a second.

Using this, a computations gives

$$\begin{aligned}\frac{1}{2}A(v+u) = A\left(\frac{1}{2}v+\frac{1}{2}u\right) &= f\left(\frac{1}{2}v+\frac{1}{2}u\right) - f(0)\\ &= \frac{1}{2}f(v)+\frac{1}{2}f(u)-\frac{1}{2}f(0)-\frac{1}{2}f(0)\\ &= \frac{1}{2}A(v)+\frac{1}{2}A(u),\end{aligned}$$

so A is additive and we just need it homogenous now. To this end we see

$$0 = A(0) = A(v-v) = A(v) + A(-v) \quad \Longrightarrow \quad A(-v) = -A(v).$$

Thus, we've reduced this to showing $A(\lambda v) = \lambda A(v)$ for $\lambda > 1$. Since A is additive, we have from induction that $A(kv) = kA(v)$ for positive integers. For a general $\lambda > 1$ decompose it as $\lambda = \lfloor\lambda\rfloor + \langle\lambda\rangle$, then finally,

$$A(\lambda v) = A(\lfloor\lambda\rfloor v + \langle\lambda\rangle v) = A(\lfloor\lambda\rfloor v) + A(\langle\lambda\rangle v) = \lfloor\lambda\rfloor A(v) + \langle\lambda\rangle A(v) = \lambda A(v),$$

as desired. □

So, if f is a bijective isometry between two real normed vector spaces it is affine: $f = A + f(0)$. If f happens to send the origin to the origin then it is linear and in particular an isomorphism. Moreover, since translating by $f(0)$ is an isometry, we see that $d(Au, Av) = d(f(u), f(v)) = d(u, v)$ and so A is a linear isometry.

That the vector spaces are real is necessary as is the surjectivity. Take complex conjugation from $\mathbb{C} \to \mathbb{C}$. This is an isometry ($|\bar{z}-\bar{w}| = |z-w|$) and is surjective, but it is not complex linear. For surjectivity, take $f : \mathbb{R} \to (\mathbb{R}^2, \|\cdot\|_\infty)$ by $f(x) = (x, |x|)$; this is an isometry but not real linear.

Denote by $\mathrm{Isom}_{\mathbf{Met}}(V)$ the isometry group of V consisting of all surjective isometries $V \to V$. Similarly, let $\mathrm{Isom}_{\mathbf{Vect}}(V)$ be the invertible linear isometry of V. The Mazur-Ulam theorem tells us we have a bijective correspondence between the sets

$$\mathrm{Isom}_{\mathbf{Met}}(V) = \mathrm{Isom}_{\mathbf{Vect}}(V) \times V \quad \text{by} \quad f \leftrightarrow (A, a)$$

However, since the composition is given by

$$(A, a)\cdot(B, b) \leftrightarrow A(B + b) + a = AB + (Ab + a) \leftrightarrow (AB, Ab + a)$$

we see the multiplication on $\mathrm{Isom}_{\mathbf{Met}}(V)$ is not just components wise, but uses the homomorphism $\varphi : \mathrm{Isom}_{\mathbf{Vect}}(V) \to \mathrm{Aut}(V)$ by $\varphi(A) = A$. So as groups we have the semi-direct product

$$\mathrm{Isom}_{\mathbf{Met}}(V) = \mathrm{Isom}_{\mathbf{Vect}}(V) \ltimes V.$$

To specialize to $\mathbb{R}^n$ with norm given by the dot product we have $\mathrm{LIsom}(\mathbb{R}^n) = O(n)$ so that

$$\mathrm{Isom}_{\mathbf{Met}}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n.$$

References

[Nic12] Bogdan Nica. The Mazur-Ulam theorem. *Expo. Math.*, 30(4):397–398, 2012.